J. Fluid Mech. (1974), *vol.* 63, *part* 1, *pp.* 81–96
Printed in Great Britain

An analytic characteristic method for steady three-dimensional isentropic flow

By ANNELIESE FROHN
DFVLR-Institut für Theoretische Gasdynamik, Aachen, Germany

(Received 14 March 1973)

The analytical characteristic method is an effective method for computing nonlinear effects in inviscid supersonic flow problems. Although only linear equations have to be solved, the results are essentially nonlinear, in the sense that the functional relations between physical state variables and space co-ordinates are nonlinear in the small perturbation parameter introduced, like the thickness ratio or incidence of a wing. This holds even for the first-order approximation of the method.

In the case of two-dimensional (plane or axisymmetric) flow the independent variables are characteristic co-ordinates, i.e. they are chosen so as to be constant along corresponding characteristic lines. The space co-ordinates are considered as dependent variables. In three dimensions there is no unique definition of a characteristic co-ordinate system, because the manifold of characteristic surfaces or bi-characteristics is larger than is necessary for defining a co-ordinate system. The success of a three-dimensional analytical characteristic method, however, depends on the proper choice of the co-ordinate system.

The present analytical characteristic method for three-dimensional flow is based on the fact that three-dimensional flow behaves locally like axisymmetric flow if it is considered in the osculating plane. The corresponding 'distance from the axis' is a function of space depending on the flow field. No change of pressure occurs normal to the osculating plane and in isentropic flow no change of speed either. Therefore no co-ordinate perturbation is performed in this normal direction. In the osculating plane the analytical characteristic method is applied locally as in axisymmetric flow. In the large the space co-ordinates are obtained by integration along the main bi-characteristics.

As an example the flow field on the suction side of a flat delta wing with subsonic leading edges is computed. As a main result one obtains shock waves in the neighbourhood of the leading edges following the expansion.

1. Introduction

The analytical characteristic method, due to Oswatitsch (1962), has proved to be very effective for computing two-dimensional (i.e. plane or axisymmetric) flow fields (Schneider 1963). It can be used also in symmetry planes of three-dimensional flow (Sun 1964) or for flow which is approximately plane or axisymmetric (Niederdrenk 1969). In the general three-dimensional case, however, the method may fail, if the characteristic co-ordinates, which are used as independent variables, are not chosen in a suitable way. Before going into the details of this problem, the basic idea and the main advantages of the two-dimensional analytical characteristic method will be pointed out. A more detailed description of it will be given in § 2.

The most important advantage of the method is that it reveals 'nonlinear' effects of the flow, although only linear equations have to be solved, and this is also true for its first-order approximation. The expression 'nonlinear' means that the functional relations between physical state variables and space co-ordinates are nonlinear in the small perturbation parameters introduced, like the incidence or thickness ratio of a wing. This is achieved by using characteristic co-ordinates as independent variables instead of space co-ordinates. Physical state variables and space co-ordinates are expanded in power series in a small perturbation parameter. By the choice of characteristic co-ordinates as independent variables one obtains also an improved representation of the characteristic lines. If only the physical state variables, as functions of the space co-ordinates, were expanded in a power series in the small perturbation parameter, as in acoustic theory, the characteristics would to any order of approximation be the same as in the undisturbed flow.

A further advantage of the method is that the existence of weak shock waves can be discovered and their location and strength can be computed to the corresponding order, as described in § 3.

The difficulty of applying the analytical characteristic method to three-dimensional flow problems arises from the fact that characteristic variables are not defined uniquely in a three-dimensional flow field. Instead of two characteristic line elements there is an infinite number of bi-characteristic line elements through every point, generating the Mach cone. Consequently there is an arbitrariness in choosing three bi-characteristic line elements as a base for a local characteristic co-ordinate system. The choice of the co-ordinate system may, however, strongly influence the results obtained by the method. Therefore the question arises of whether there are bi-characteristics which are in some way more relevant than the others, and which are therefore suitable as a base for a local characteristic co-ordinate system. The answer to this question is given by considering the flow locally in the osculating plane of the streamline. In this plane the equations governing the flow have the same form as the equations for axisymmetric flow. The only difference is that the corresponding 'distance from the axis' depends on the flow field. In the binormal direction no pressure change occurs and in isentropic flow no change of speed either. This means that the changes in the physical state variables occur mainly in the osculating plane of the streamline. The two bi-characteristics lying in this plane are called main bi-characteristics.

As a result of this consideration the two main bi-characteristics and the binormal are chosen as a base for the local co-ordinate system instead of three bi-characteristics. At every point the analytical characteristic method for axisymmetric flow is applied locally in the osculating plane of the streamline. No co-ordinate perturbation is performed in the binormal direction.

In the large the co-ordinate system is replaced by the zeroth-order space co-ordinates, which are related to the characteristic variables by the Monge equation. Thus, the higher order space co-ordinates are given in terms of line integrals along the main bi-characteristics as functions of the zeroth-order space co-ordinates. For the case of isentropic flow the first-order approximation will give sufficiently good results. There is not much sense in considering a second-order approximation for the space co-ordinates without considering entropy changes as well. Only the second-order approximation to the physical state variables would be in no contradiction to the assumption of isentropy.

2. The analytical characteristic method for two-dimensional isentropic flow

For plane isentropic flow the equations in characteristic variables l and m have a simple form. The compatibility conditions are

$$\frac{(M^2-1)^{\frac{1}{2}}}{q}\frac{\partial q}{\partial l}-\frac{\partial \theta}{\partial l}=0, \quad \frac{(M^2-1)^{\frac{1}{2}}}{q}\frac{\partial q}{\partial m}+\frac{\partial \theta}{\partial m}=0. \tag{2.1}$$

The characteristic equations are

$$\frac{\partial x}{\partial l}-\tan(\theta+\alpha)\frac{\partial y}{\partial l}=0, \quad \frac{\partial x}{\partial m}-\tan(\theta-\alpha)\frac{\partial y}{\partial m}=0. \tag{2.2}$$

Here q denotes the flow speed, M the local Mach number, α the local Mach angle and θ the angle of the streamline relative to some constant direction, usually the direction of the undisturbed flow. Further, x and y are Cartesian co-ordinates in the flow field.

Equation (2.1) can be integrated exactly, but in order to satisfy the boundary conditions one has to know the solution of (2.2), which again depends on the solution of (2.1). Because of this difficulty the dependent variables are expanded in a power series in a small perturbation parameter τ. The undisturbed flow is assumed to be uniform, with speed q_0, and $\theta_0 = 0$:

$$q = q_0+\tau q_1(l,m)+\ldots, \quad \theta = \tau\theta_1(l,m)+\ldots, \tag{2.3}$$

$$x = x_0(l,m)+\tau x_1(l,m)+\ldots, \quad y = y_0(l,m)+\tau x_1(l,m)+\ldots. \tag{2.4}$$

First x_0 and y_0 are determined from (2.2). They are then used to satisfy the boundary conditions for q_1 and θ_1 are known, the first-order terms x_1 and y_1 can be determined from (2.2). In this alternating way higher order terms can also be computed.

The solution when in the form of (2.3) and (2.4) defines functional relations $q = q(x,y;\tau)$ and $\theta = \theta(x,y;\tau)$ which in general cannot be given explicitly. It can be seen, however, from (2.3) and (2.4) that these functional relations are non-linear in τ, even if only first-order terms of q, θ, x and y are considered.

In addition, the functions $q(x,y;\tau)$ and $\theta(x,y;\tau)$ are in general not uniquely determined in the whole flow field, because there are usually regions in which (2.4) cannot be inverted uniquely. In order to obtain uniquely defined solutions $q(x,y;\tau)$ and $\theta(x,y;\tau)$ one has to assume the existence of shock waves in the regions of non-uniqueness.

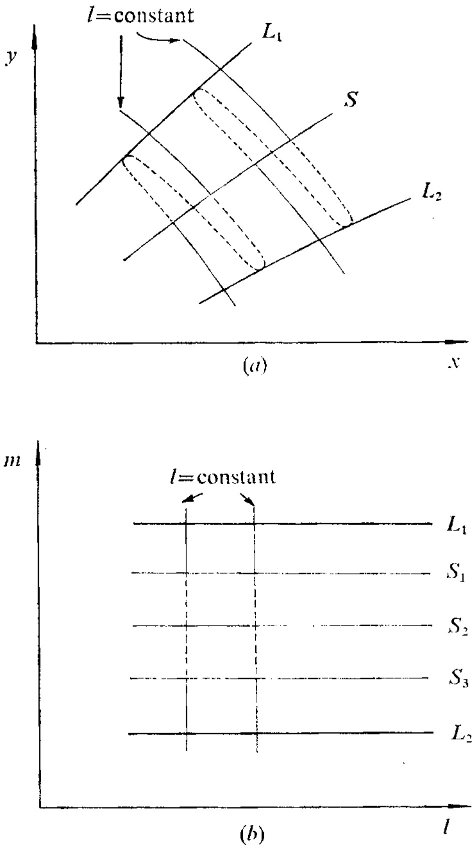

FIGURE 1. Region of folding in (a) the physical plane and (b) the characteristic plane.

3. Weak shock waves

A region of the flow field in which the functions $x(l, m)$ and $y(l, m)$ cannot be inverted uniquely is called a region of folding. Such a region is multiply covered by characteristics. Since the physical state variables are functions of l and m, they are not unique functions of x and y in a region of folding.

In order to obtain a uniquely defined solution for the physical state variables one has to assume a shock wave in a region of folding. To explain this, a characteristic plane is introduced in which l and m are Cartesian co-ordinates, whereas the plane with Cartesian co-ordinates x and y is called the physical plane. The region of folding is shown in figure 1 (a). It is bounded by the curves L_1 and L_2. The corresponding region in the characteristic plane is shown in figure 1 (b). Three curves S_1, S_2 and S_3 in the characteristic plane are related to every curve S in the physical plane. The region between S_1 and S_3 is indicated in the physical plane by the dotted part of the lines l = constant. If the region between S_1 and S_3 is cut off, one obtains a uniquely defined solution for the physical state variables in the physical plane. This solution changes discontinuously across the curve S, from the values which are given on the curve S_1 to the values on S_3.

Thus S represents a shock wave, if it is chosen in such a way that the discontinuity satisfies the shock conditions across S. The numerical method for computing the location of the shock curve is described in § 6 for the case of a conical flow field.

4. Basic equations

4.1. *Equations in intrinsic form*

In Cartesian co-ordinates (x, y, z) the differential equations for steady flow of an inviscid ideal gas are, in terms of the density ρ, velocity vector $\mathbf{v}$, pressure p and entropy S,

$$\nabla(\rho\mathbf{v}) = 0 \quad \text{(continuity)}, \tag{4.1}$$

$$(\mathbf{v}.\nabla)\,\mathbf{v}+\rho^{-1}\nabla p = 0 \quad \text{(momentum)}, \tag{4.2}$$

$$(\mathbf{v}.\nabla)\,S = 0 \quad \text{(energy)} \tag{4.3}$$

with the notation

$$\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z).$$

Define $\mathbf{s}, \mathbf{n}$ and $\mathbf{b}$ at every point as unit vectors in the direction of the streamline, its main normal and its binormal. The derivatives in these directions will be written briefly as

$$\mathbf{s}.\nabla = \partial/\partial s, \quad \mathbf{n}.\nabla = \partial/\partial n, \quad \mathbf{b}.\nabla = \partial/\partial b.$$

By multipling the vector equation (2.2) by $\mathbf{s}$, $\mathbf{n}$ and $\mathbf{b}$, and introducing the flow speed q for which the relation $\mathbf{v} = q\mathbf{s}$ holds, one obtains the following scalar equations:

$$q\frac{\partial q}{\partial s}+\frac{1}{\rho}\frac{\partial p}{\partial s} = 0, \tag{4.4}$$

$$q^2\left(\mathbf{n}.\frac{\partial \mathbf{s}}{\partial s}\right)+\frac{1}{\rho}\frac{\partial p}{\partial n} = 0, \tag{4.5}$$

$$\partial p/\partial b = 0. \tag{4.6}$$

Assuming isentropic isoenergetic flow, one can show with the help of the energy equation that (4.4) can be replaced by the more general relation

$$q\,dq+\rho^{-1}\,dp = 0, \tag{4.7}$$

which is used to eliminate the pressure p. The density ρ is eliminated by the relation

$$d\rho = a^{-2}\,dp, \tag{4.8}$$

which holds for isentropic flow. Here a denotes the speed of sound.

If (4.7) and (4.8) are inserted into (4.1), (4.5) and (4.6) one obtains the following equations for the unknown velocity:

$$\left(1-\frac{q^2}{a^2}\right)\frac{\partial q}{\partial s}+q\left(\mathbf{n}.\frac{\partial \mathbf{s}}{\partial n}\right)+q\left(\mathbf{b}.\frac{\partial \mathbf{s}}{\partial b}\right) = 0, \tag{4.9}$$

$$q\mathbf{n}.\frac{\partial \mathbf{s}}{\partial s}-\frac{\partial q}{\partial n} = 0, \tag{4.10}$$

$$\partial q/\partial b = 0. \tag{4.11}$$

The speed of sound a will be expressed in terms of q using the energy equation in the form

$$a^2+\tfrac{1}{2}(\kappa-1)\,q^2 = \text{constant},$$

where κ denotes the ratio of specific heats.

Equations (4.9)–(4.11) reveal the relation of general three-dimensional flow to plane or axisymmetric flow. The term $\mathbf{b}\,.\,\partial\mathbf{s}/\partial b$ is zero for plane flow. For axisymmetric flow it is inversely proportional to the distance from the axis. This shows that the surface elements which are generated by the vectors $\mathbf{s}$ and $\mathbf{n}$ play locally the role of the symmetry plane of two-dimensional flow. Further, changes in the quantities q, p and ρ occur only in these surface elements because of (4.6), (4.8) and (4.11). Therefore the two families of bi-characteristics, which are composed of line elements lying in these surface elements, are called main bi-characteristics.

4.2. *Relations along the main bi-characteristics*

The unit vectors in the directions of the main bi-characteristics are denoted by $\mathbf{l}$ and $\mathbf{m}$. The derivatives in these directions are

$$\mathbf{l}\,.\,\nabla = \partial/\partial l, \quad \mathbf{m}\,.\,\nabla = \partial/\partial m.$$

In terms of the Mach angle α the vectors $\mathbf{l}$ and $\mathbf{m}$ are related by definition to the vectors $\mathbf{s}$ and $\mathbf{n}$ by

$$\mathbf{l} = \mathbf{s}\cos\alpha+\mathbf{n}\sin\alpha, \quad \mathbf{m} = \mathbf{s}\cos\alpha-\mathbf{n}\sin\alpha. \tag{4.12}$$

The local Mach number $M = q/a$ is related to the Mach angle by

$$\sin\alpha = 1/M.$$

In terms of the derivatives $\partial/\partial l$ and $\partial/\partial m$ equations (4.9)–(4.11) become

$$\left.\begin{aligned} (M^2-1)^{\frac{1}{2}}\frac{\partial q}{\partial l}-q\left(\mathbf{n}\,.\,\frac{\partial\mathbf{s}}{\partial l}\right) &= a\left(\mathbf{b}\,.\,\frac{\partial\mathbf{s}}{\partial b}\right),\\ (M^2-1)^{\frac{1}{2}}\frac{\partial q}{\partial m}+q\left(\mathbf{n}\,.\,\frac{\partial\mathbf{s}}{\partial m}\right) &= a\left(\mathbf{b}\,.\,\frac{\partial\mathbf{s}}{\partial b}\right),\\ 0 &= \partial q/\partial b. \end{aligned}\right\} \tag{4.13}$$

4.3. *Monge equation*

For the space co-ordinates the Monge equation is valid along any bi-characteristic. In Cartesian co-ordinates this equation is

$$\begin{aligned}(a^2-v^2-w^2)\,dx^2+(a^2-u^2-w^2)\,dy^2+(a^2-u^2-v^2)\,dz^2&\\ +2vw\,dy\,dz+2uw\,dx\,dz+2uv\,dx\,dy &= 0.\end{aligned} \tag{4.14}$$

Using this equation along the two main bi-characteristic directions one obtains two equations, which will serve to determine the space co-ordinates. They will be completed by postulating that no co-ordinate perturbation is performed in the $\mathbf{b}$ direction.

4.4. *Equations in Cartesian co-ordinates*

In the following (4.9)–(4.11) will be used also in terms of Cartesian co-ordinates. The components of $\mathbf{v}$ in the x, y and z direction are denoted by u, v and w respectively. In terms of the velocity components (4.9)–(4.11) become

$$(u^2-c^2)\frac{\partial u}{\partial x}+(v^2-c^2)\frac{\partial v}{\partial x}+(w^2-c^2)\frac{\partial w}{\partial x}$$
$$+vw\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+uw\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+uv\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=0, \qquad (4.15)$$

$$\nabla\times\mathbf{v}=0. \qquad (4.16)$$

Because of (4.17) a velocity potential ϕ can be introduced:

$$\mathbf{v}=\nabla\phi.$$

In terms of ϕ equation (4.16) becomes

$$(u^2-c^2)\frac{\partial^2\phi}{\partial x^2}+(v^2-c^2)\frac{\partial^2\phi}{\partial y^2}+(w^2-c^2)\frac{\partial^2\phi}{\partial z^2}+2vw\frac{\partial^2\phi}{\partial y\,\partial z}+2uw\frac{\partial^2\phi}{\partial x\,\partial z}+2uv\frac{\partial^2\phi}{\partial x\,\partial y}=0. \qquad (4.17)$$

5. Dependent and independent variables

In the following the dependent variables are u, v and w, representing the physical state variables, and the Cartesian space co-ordinates (x, y, z).

The independent variables are denoted by x_0, y_0 and z_0. They are defined to be equal to the Cartesian space co-ordinates at the zeroth order of approximation. The relation of the variables x_0, y_0 and z_0 to the local characteristic co-ordinates is given by

$$\left.\begin{aligned}\left(\frac{\partial x_0}{\partial l}\right)^2-\beta^2\left(\frac{\partial y_0}{\partial l}\right)^2-\beta^2\left(\frac{\partial z_0}{\partial l}\right)^2&=0,\\ \left(\frac{\partial x_0}{\partial m}\right)^2-\beta^2\left(\frac{\partial y_0}{\partial m}\right)^2-\beta^2\left(\frac{\partial z_0}{\partial m}\right)^2&=0,\end{aligned}\right\} \qquad (5.1)$$

where β^2 is short for M_0^2-1, M_0 denoting the Mach number of the undisturbed flow.

Relations (5.1) are at the zeroth order of approximation identical to the Monge equations and are therefore at this order of approximation valid along any bi-characteristic. At higher orders of approximation, however, they hold only along the main bi-characteristics. Nevertheless the space with Cartesian co-ordinates (x_0, y_0, z_0) is called the 'characteristic space'.

6. Zeroth-order of approximation

The dependent variables are expanded in a power series with respect to a perturbation parameter τ:

$$\left.\begin{aligned} x &= x_0+\tau x_1(x_0,y_0,z_0)+\dots,\\ y &= y_0+\tau y_1(x_0,y_0,z_0)+\dots,\\ z &= z_0+\tau z_1(x_0,y_0,z_0)+\dots, \end{aligned}\right\} \tag{6.1}$$

$$\mathbf{v} = v_0+\tau \mathbf{v}_1(x_0,y_0,z_0)+\dots. \tag{6.2}$$

The zeroth-order velocity is determined by the undisturbed flow, which is assumed to be a uniform flow in the x direction with velocity $\mathbf{v}_0 = (u_0, 0, 0)$, sound speed a_0 and Mach number $M_0 = u_0/a_0$. The zeroth-order space co-ordinates are given by the definition of the independent variables. The zeroth-order term of $\mathbf{s}$ is given by

$$\mathbf{s}_0 = (1,0,0).$$

The zeroth-order terms of the vectors $\mathbf{n}$ and $\mathbf{b}$, however, contain the unknown first-order terms v_1 and w_1. Since $\mathbf{n}$ is defined as a unit vector in the direction of $\partial\mathbf{s}/\partial s$, one obtains

$$\mathbf{n}_0 = (0,\gamma,\gamma'), \tag{6.3}$$

with

$$\left.\begin{aligned} \gamma &= \frac{\partial v_1/\partial x_0}{[(\partial v_1/\partial x_0)^2+(\partial w_1/\partial x_0)^2]^{\frac{1}{2}}},\\ \gamma' &= \frac{\partial w_1/\partial x_0}{[(\partial v_1/\partial x_0)^2+(\partial w_1/\partial x_0)^2]^{\frac{1}{2}}}, \end{aligned}\right\} \tag{6.4}$$

and for $\mathbf{b}_0$, which is orthogonal to $\mathbf{n}_0$ and $\mathbf{s}_0$,

$$\mathbf{b}_0 = (0,-\gamma,\gamma'). \tag{6.5}$$

Thus

$$\mathbf{l}_0 = \left(\frac{\beta}{M_0},\frac{\gamma}{M_0},\frac{\gamma'}{M_0}\right), \quad \mathbf{m}_0 = \left(\frac{\beta}{M_0},\frac{-\gamma}{M_0},\frac{-\gamma'}{M_0}\right). \tag{6.6}$$

Since $\mathbf{l}_0$ and $\mathbf{m}_0$ are needed to determine x_1, y_1 and z_1, one has to compute $\mathbf{v}_1$ first.

7. First-order of approximation

By inserting (6.2) into (4.15) and (4.16) and observing that at zeroth order

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial x_0}, \quad \frac{\partial}{\partial y}=\frac{\partial}{\partial y_0}, \quad \frac{\partial}{\partial z}=\frac{\partial}{\partial z_0}$$

one obtains the equations

$$\beta^2\frac{\partial u_1}{\partial x_0}-\frac{\partial v_1}{\partial y_0}-\frac{\partial w_1}{\partial z_0} = 0, \tag{7.1}$$

$$\nabla_0\times v_1 = 0. \qquad (7.2)\dagger$$

† The symbols $\nabla_0\times$ and ∇_0 denote the operators $\nabla\times$ and ∇ with respect to (x_0,y_0,z_0).

In terms of a first-order potential ϕ_1 which is defined as

$$\mathbf{v}_1 = \nabla_0 \phi_1, \tag{7.3}†$$

equation (7.1) becomes
$$\beta^2 \frac{\partial^2 \phi_1}{\partial x_0^2} - \frac{\partial^2 \phi_1}{\partial y_0^2} - \frac{\partial^2 \phi_1}{\partial z_0^2} = 0. \tag{7.4}$$

With $\mathbf{v}_1$ computed from (7.3) and (4.4), $\mathbf{l}_0$ and $\mathbf{m}_0$ are given by (6.6) and (6.4).

In the first-order approximation to the Monge equations the derivatives $\partial/\partial l$ and $\partial/\partial m$ are equal to $\mathbf{l}_0 . \nabla_0$ and $\mathbf{m}_0 . \nabla_0$ respectively; they are denoted by $\partial/\partial l_0$ and $\partial/\partial m_0$.†

$$a_0^2 \left(\frac{\partial x_0}{\partial l_0}\frac{\partial x_1}{\partial l_0} - \beta^2 \frac{\partial y_0}{\partial l_0}\frac{\partial y_1}{\partial l_0} - \beta^2 \frac{\partial z_0}{\partial l_0}\frac{\partial z_1}{\partial l_0} \right) + a_0 a_1 \left(\frac{\partial x_0}{\partial l_0} \right)^2$$
$$+ (a_0 a_1 - u_0 u_1) \left[\left(\frac{\partial y_0}{\partial l_0} \right)^2 + \left(\frac{\partial z_0}{\partial l_0} \right)^2 \right] + u_0 v_0 \frac{\partial x_0}{\partial l_0}\frac{\partial y_0}{\partial l_0} + u_0 w_1 \frac{\partial x_0}{\partial l_0}\frac{\partial z_0}{\partial l_0} = 0, \tag{7.5a}$$

$$a_0^2 \left(\frac{\partial x_0}{\partial m_0}\frac{\partial x_1}{\partial m_0} - \beta^2 \frac{\partial y_0}{\partial m_0}\frac{\partial y_1}{\partial m_0} - \beta^2 \frac{\partial z_0}{\partial m_0}\frac{\partial z_1}{\partial m_0} \right) + a_0 a_1 \left(\frac{\partial x_0}{\partial m_0} \right)^2$$
$$+ (a_0 a_1 - u_0 u_1) \left[\left(\frac{\partial y_0}{\partial m_0} \right)^2 + \left(\frac{\partial z_0}{\partial m_0} \right)^2 \right] + u_0 v_1 \frac{\partial x_0}{\partial m_0}\frac{\partial y_0}{\partial m_0} + u_0 w_1 \frac{\partial x_0}{\partial m_0}\frac{\partial y_0}{\partial m_0} = 0. \tag{7.5b}$$

Using (6.6) the first-order approximation to the Monge equations becomes

$$\left.\begin{aligned} \frac{\partial x_1}{\partial l_0} - \gamma\beta \frac{\partial y_1}{\partial l_0} - \gamma'\beta \frac{\partial z_1}{\partial l_0} &= \frac{\kappa+1}{2} \frac{M_0^3}{\beta} \frac{u_1}{u_0} - \frac{M_0^2}{u_0} \frac{\partial \phi_1}{\partial l_0}, \\ \frac{\partial x_1}{\partial m_0} + \gamma\beta \frac{\partial y_1}{\partial m_0} + \gamma'\beta \frac{\partial z_1}{\partial m_0} &= \frac{\kappa+1}{2} \frac{M_0^3}{\beta} \frac{u_1}{u_0} - \frac{M_0^2}{u_0} \frac{\partial \phi_1}{\partial m_0}. \end{aligned}\right\} \tag{7.6}$$

An additional equation for the first-order space co-ordinates is obtained by postulating that no co-ordinate perturbation is performed in $\mathbf{b}$ direction:

$$\gamma' y_1 - \gamma z_1 = 0. \tag{7.7}$$

Using this equation one can integrate (7.6):

$$\left.\begin{aligned} x_1 - \gamma\beta y_1 - \gamma'\beta z_1 &= \frac{\kappa+1}{2} \frac{M_0^3}{\beta} \int \frac{u_1}{u_0} dl_0 - \frac{M_0^2}{u_0} \phi_1, \\ x_1 + \gamma\beta y_1 + \gamma'\beta z_1 &= \frac{\kappa+1}{2} \frac{M_0^3}{\beta} \int \frac{u_1}{u_0} dm_0 - \frac{M_0^2}{u_0} \phi_1. \end{aligned}\right\} \tag{7.8}$$

Here the integrals are to be taken along the main bi-characteristics.

8. Discussion of the first order of approximation

The combined solutions of (7.1), (7.2) and (7.8) represent the result of the first-order approximation of the present method:

$$\mathbf{u}(x,y,z) = \mathbf{u}_0 + \tau \mathbf{u}_1(x_0, y_0, z_0), \tag{8.1}$$

$$\left.\begin{aligned} x &= x_0 + \tau x_1(x_0, y_0, z_0), \\ y &= y_0 + \tau y_1(x_0, y_0, z_0), \\ z &= z_0 + \tau z_1(x_0, y_0, z_0). \end{aligned}\right\} \tag{8.2}$$

† See footnote on p. 88.

Consider first the case when $u_1(x_0, y_0, z_0)$ can be approximated by the first two terms of a Taylor series along $\mathbf{x}_1 = (x_1, y_1, z_1)$:

$$\mathbf{u}_1(x_0, y_0, z_0) = \mathbf{u}_1(x, y, z) - \tau(\mathbf{x}_1 . \nabla)\,\mathbf{u}_1(x, y, z). \tag{8.3}$$

By inserting (8.3) into (8.1) one obtains a solution which differs from the first-order solution of acoustic theory by the second-order term

$$-\tau^2(\mathbf{x}_1 . \nabla)\,\mathbf{u}_1(x, y, z). \tag{8.4}$$

It is, however, not the purpose of the present method to give just a second-order effect. This could have been done earlier using a higher order acoustic theory. In the present method interest is concentrated on those regions in which (8.3) does not hold. In this case the nonlinear functional relation $\mathbf{u}_1(x, y, z; \tau)$ is such that it cannot be expanded in a power series in τ. Thus, one could not obtain solutions depending on τ in this way from acoustic theory to any order.

Examples of regions in which (8.3) does not hold are the following.

(i) The neighbourhood of subsonic leading edges of wings, where $\mathbf{u}(x_0, y_0, z_0)$ becomes infinite usually.

(ii) The neighbourhood of surfaces which correspond to shock waves. As described in § 3 there are two or more surfaces in characteristic space corresponding to a shock surface in physical space. In the neighbourhood of these surfaces there is usually a surface on which the normal gradient of the normal velocity component is infinite or even with infinite values of the velocity components.

9. Second order of approximation

The equations for the second-order velocity components are given by the second-order terms of (4.15) and (4.16), observing that at first order the relations between the derivatives in physical and characteristic space are

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_0} - \frac{\partial \mathbf{x}_1}{\partial x_0} . \nabla_0, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial y_0} - \frac{\partial \mathbf{x}_1}{\partial y_0} . \nabla_0, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z_0} - \frac{\partial \mathbf{x}_1}{\partial z_0} . \nabla_0. \tag{9.1}$$

By considering further (7.1) and (7.2) one can write the second-order equations in the form

$$\beta^2 \frac{\partial}{\partial x_0}(u_2 - \mathbf{x}_1 . \nabla_0 u_1) - \frac{\partial}{\partial y_0}(v_2 - \mathbf{x}_1 . \nabla_0 v_1) - \frac{\partial}{\partial z_0}(w_2 - \mathbf{x}_1 . \nabla_0 w_1)$$
$$= \frac{1}{u_0}\frac{\partial}{\partial x_0}\left(\frac{\kappa + 1}{2} M_0^2 u_1^2 + v_1^2 + w_1^2\right), \tag{9.2}$$

$$\nabla_0 \times (\mathbf{u}_2 - (\mathbf{x}_0 . \nabla_0)\,\mathbf{u}_1) = 0. \tag{9.3}$$

In (9.2) essential terms would cancel if the relations (7.8) were inserted. It is, however, more convenient to use the equations in the forms (9.2) and (9.3). Using the components of the vector

$$\mathbf{u}_2^* = \mathbf{u}_2 - (\mathbf{x}_1 . \nabla_0)\,\mathbf{u}_1 \tag{9.4}$$

as new dependent variables one has to solve the same equations as in second-order acoustic theory. Because of (9.3) one can introduce a potential for the vector $\mathbf{u}_2^*$, whereas there is usually no potential for the vector $\mathbf{u}_2$.

For all higher order terms of the velocity, component equations of the form of (9.2) and (9.3) have to be solved.

In order to obtain the second-order terms of the Monge equations one has to express the first-order parts of the derivatives $\partial/\partial l$ and $\partial/\partial m$ in terms of derivatives with respect to x_0, y_0 and z_0. First the terms $\mathbf{l}_1$ and $\mathbf{m}_1$ of the vectors $\mathbf{l}$ and $\mathbf{m}$ are computed using the results of (9.3) and (9.2). Using the relations (9.1) and (7.8) one obtains

$$\left.\begin{aligned}\frac{\partial}{\partial l} &= \left(1+\tau\left(\mathbf{l}_1-\frac{\partial \mathbf{x}_1}{\partial l_0}\right).\mathbf{l}_0\right)\frac{\partial}{\partial l_0}+\tau\left(\mathbf{l}_1-\frac{\partial \mathbf{x}_1}{\partial l_0}\right).\mathbf{b}_0\frac{\partial}{\partial b_0},\\ \frac{\partial}{\partial m} &= \left(1+\tau\left(\mathbf{m}_1-\frac{\partial \mathbf{x}_1}{m_0}\right).\mathbf{m}_0\right)\frac{\partial}{\partial m_0}+\tau\left(\mathbf{m}_1-\frac{\partial \mathbf{x}_1}{\partial m_0}\right).\mathbf{b}_0\frac{\partial}{\partial b_0}.\end{aligned}\right\} \tag{9.5}$$

Thus the second-order terms of the Monge equations have the same form as (7.8):

$$\left.\begin{aligned}x_2-\gamma y_2-\gamma' z_2 &= F(\mathbf{x}_1,\mathbf{u}_1,\mathbf{u}_2)\,dl_0,\\ x_2+\gamma y_2+\gamma' z_2 &= G(\mathbf{x}_1,\mathbf{u}_1,\mathbf{u}_2)\,dm_0.\end{aligned}\right\} \tag{9.6}$$

The right-hand sides of (9.6) contain only the known functions F and G, which are obtained by expanding the Monge equations and using the relations (9.5).

All higher order terms of the space co-ordinates will be given by relations of the form (9.6).

10. Example: upper side of a conical flat delta plate with subsonic leading edges

Conical flow problems are well suited for studying general three-dimensional flow problems, because conical symmetry does not change the three-dimensional nature of the flow, as plane or axial symmetry does. The problem of nonlinear conical flow has been considered by many authors. A survey of methods and results has been given by Bulakh (1970).

The method of computing the location and strength of weak shock waves by solving linear equations was described first by Lighthill (1949). For the case of a wing with subsonic leading edges and arbitrary cross-section the present method gives at first-order formulae for the location and strength of the head wave similar to those given by Lighthill; the only difference is in the boundary condition of the starting linearized solution, because the present method is applied in the whole flow field and not only in the neighbourhood of the shock wave. The span of a wing in characteristic space is usually different from the actual span in physical space. This may have an essential effect on the location and strength of the shock wave, e.g. if the leading edges are nearly sonic.

A much more complicated situation occurs in the case of inner shock waves, such as those at the leading edges on the upper side of an inclined flat delta wing with nearly sonic leading edges. Except in the case of a head wave, no *a priori* assumptions can be made, like assuming the shock shape to be approximately a circular cone or the flow in front of the shock to be uniform. In the following example not even the existence of a shock was presumed from the beginning.

If a characteristic method is applied to a conical flow problem, it has to be treated first as a three-dimensional problem, because the conical gasdynamic equations, which contain only two independent variables, are of elliptic type. Since in conical flow the physical quantities are constant along straight lines through the origin, the following conical co-ordinate systems are used in characteristic space and in physical space:

$$\left.\begin{aligned} \xi_0 = x_0, \quad \eta_0 = y_0/x_0, \quad \zeta_0 = z_0/x_0 \quad &\text{in characteristic space,} \\ \xi = x, \quad \eta = y/x, \quad \zeta = z/x \quad &\text{in physical space.} \end{aligned}\right\} \tag{10.1}$$

The physical quantities are functions of η_0 and ζ_0 only. The potential ϕ has the form

$$\phi = \xi_0 \bar{\phi}(\eta_0, \zeta_0).$$

The equation for the function $\bar{\phi}$, which is called the conical potential, is at first order, according to (7.3),

$$(\eta_0^2 - 1)\frac{\partial^2 \bar{\phi}_1}{\partial \eta_0^2} + (\zeta_0^2 - 1)\frac{\partial^2 \bar{\phi}_1}{\partial \zeta_0^2} + 2\eta_0 \zeta_0 \frac{\partial^2 \bar{\phi}_1}{\partial \eta_0 \partial \zeta_0} = 0. \tag{10.2}$$

The linearized solution for the inclined conical delta plate has been given by Robinson (1946). It is given here in a similar co-ordinate system.

First a Lorentz transform is applied, such that the inclined wing is located in the symmetry plane of the undisturbed Mach cone. The transformed co-ordinates are denoted by $(\bar{\xi}_0, \bar{\eta}_0, \bar{\zeta}_0)$. A new conical co-ordinate system is introduced, leaving $\bar{\xi}_0$ unchanged:

$$\bar{\eta}_0 = \beta^{-1} \cos\varphi \cos\psi, \quad \bar{\zeta}_0 = \beta^{-1}(1 - k_0'^2 \sin^2\varphi)^{\frac{1}{2}} \sin\psi, \tag{10.3}$$

where k_0 denotes the zeroth-order span of the wing and $k_0'^2 = 1 - k_0^2$. In these co-ordinates the conical velocity potential for the flat delta plate at an inclination τ is found to be

$$\bar{\phi}_1 = \frac{u_0 k_0^2 \tau}{\beta E(k_0')} \cos\varphi \cos\psi \int_0^{\varphi} \frac{(1 - k_0'^2 \sin^2\sigma)^{\frac{1}{2}}}{\cos^2\sigma} d\sigma, \tag{10.4}$$

where $E(k_0')$ denotes the complete normal elliptic integral of the second kind. The velocity components obtained from the potential are

$$u_1 = \frac{u_0 k_0^2 \tau}{E(k_0')} \frac{\cos\psi \sin\varphi (1 - k_0'^2 \sin^2\varphi)^{\frac{1}{2}}}{k_0^2 \cos^2\psi + k_0'^2 \cos^2\varphi}, \tag{10.5a}$$

$$v_1 = \frac{u_0 k_0^2 \tau}{E(k_0')} \frac{k_0'^2 \cos\varphi \sin\varphi (1 - k_0'^2 \sin\varphi)^{\frac{1}{2}}}{k_0^2(k_0^2 \cos^2\psi + k_0'^2 \cos^2\varphi)} - \frac{E(\varphi, k_0')}{k_0^2}, \tag{10.5b}$$

$$w_1 = \frac{u_0 k_0^2 \tau}{E(k_0')} \frac{\cos\psi \sin\psi \sin\varphi}{k_0^2 \cos^2\psi + k_0'^2 \cos\varphi}. \tag{10.5c}$$

Since at the leading edges $\cos\varphi = 0$ and $\cos\psi = 0$ the velocity components have singularities there, corresponding to the fact that the expansion around the leading edges is accompanied by a rather higher flow speed.

The first-order space co-ordinates have the form $\bar{\xi}_0 f(\bar{\eta}_0, \bar{\zeta}_0)$, otherwise there is no conical symmetry in physical space. The first-order conical co-ordinates are related to the first-order Cartesian co-ordinates by

$$\bar{\xi}_1 = x_1, \quad \eta_1 = \frac{y_1 - \bar{\eta}_0 x_1}{\bar{\xi}_0}, \quad \bar{\zeta}_1 = \frac{z_1 - \bar{\zeta}_0 x_1}{\bar{\xi}_0}. \tag{10.6}$$

Only η_1 and ζ_1 are relevant; they are functions of $\bar{\eta}_0$ and $\bar{\zeta}_0$ only. ξ_1 can be eliminated by the assumption of no co-ordinate perturbation in the binomial direction

$$\xi_1 = -(\gamma'\eta_1 - \gamma\zeta_1)/(\gamma'\bar{\eta}_0 - \gamma\bar{\zeta}_0), \tag{10.7}$$

however, special considerations have to be made in the neighbourhood of those points or lines on which $\gamma'\bar{\eta}_0 - \gamma\bar{\zeta}_0 = 0$.

The two unknown functions η_1 and ζ_1 are then obtained from the Monge equations, observing that for functions depending on $\bar{\eta}_0$ and $\bar{\zeta}_0$ only the derivatives $\partial/\partial l_0$ and $\partial/\partial m_0$ are given by

$$\left.\begin{aligned} \frac{\partial}{\partial l_0} &= (\gamma - \beta\bar{\eta}_0)\frac{\partial}{\partial \bar{\eta}_0} + (\gamma' - \beta\bar{\zeta}_0)\frac{\partial}{\partial \bar{\zeta}_0}, \\ \frac{\partial}{\partial m_0} &= -(\gamma + \beta\bar{\eta}_0)\frac{\partial}{\partial \bar{\eta}_0} - (\gamma' + \beta\bar{\zeta}_0)\frac{\partial}{\partial \bar{\zeta}_0}. \end{aligned}\right\} \tag{10.8}$$

The integration of the Monge equations has been carried out numerically, because their right-hand sides and the functions γ and γ', derived from (10.5), are complicated functions of $\bar{\eta}_0$ and $\bar{\zeta}_0$.

Only along the wing surface can they be integrated analytically. Since at the wing surface $\bar{\eta}$ is equal zero and from (10.5) one finds $\gamma = 0$, the derivatives (10.8) reduce to derivatives with respect to $\bar{\zeta}$ only. Therefore one cannot request that ζ_1 at the wing surface be zero. This means that the mapping from characteristic space into physical space changes the span of the wing. One finds that the span becomes smaller on the upper side and larger on the lower side, because u_1 is positive on the upper side, whereas it is negative on the lower side. Therefore on the upper side one has to choose the parameter k_0 larger than the given span k of the wing, whereas on the lower side one has to choose k_0 smaller than k. Integration of the Monge equation along the wing surface gives the relation

$$k = k_0 - \frac{(\kappa+1)M_0^4}{\beta^3}\frac{k_0^2}{k_0' E(k_0')}\pi.$$

If k reaches values close to unity, k_0 has to be chosen larger than one. The solution, which is obtained by formally setting $k_0 > 1$ in (10.5), is not equal to the acoustic solution for a wing with supersonic leading edges, although the zeroth-order leading edges are located outside the undisturbed Mach cone. The solution becomes double valued in the region outside the undisturbed Mach cone formed by the wing, the Mach cone and its tangents through the leading edges. This can be seen from the definition of the co-ordinates (10.3), which are double valued in this region. But since the co-ordinate perturbation has also two values in every point of this part of characteristic space, one obtains a uniquely defined solution in physical space, except in a small region, in which a shock wave is postulated,

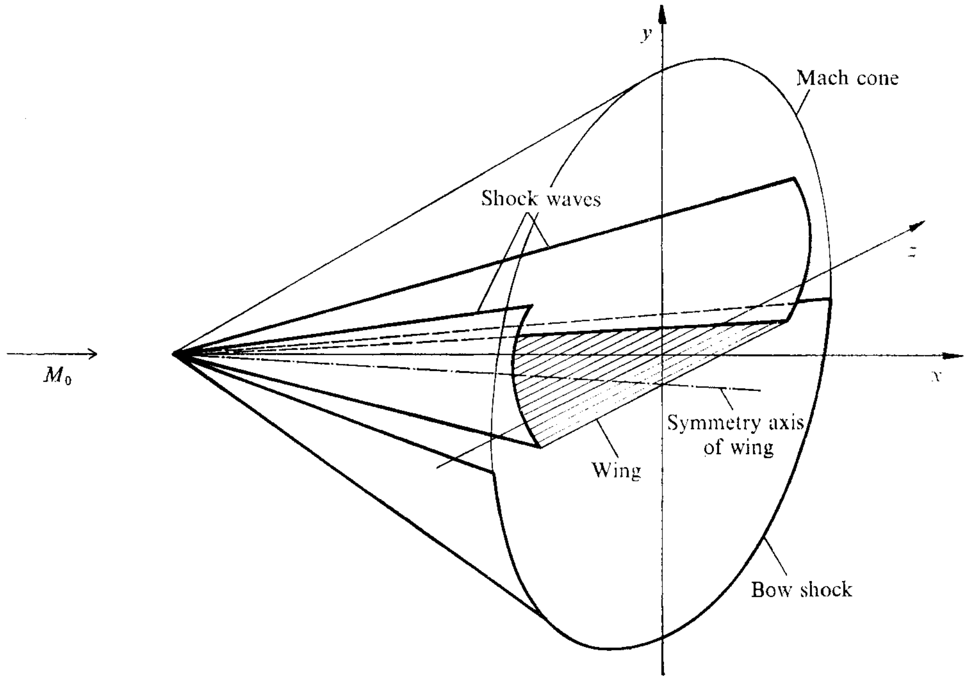

FIGURE 2. Wing with shock waves and Mach cone in Cartesian co-ordinate system. Mach number of undisturbed flow $M_0 = 2$, wing span $k/\beta = 0{\cdot}5$, angle of attack $\tau = 0{\cdot}06$ rad.

as described in §3, in order to obtain a unique solution. It is known that a shock runs upward from the leading edge, owing to overexpansion about the leading edge. This shock has been considered locally in the neighbourhood of the edge by Fraenkel & Watson (1964). The local solution was given first by Guderley (1954) for the case of a plane flow over an inclined flat plate. The shock wave does not begin exactly at the leading edge, but at some very small distance from the edge; however, within the accuracy of the present method it may be assumed to start at the leading edge.

The location of the shock is then computed by the following method. Assume one point of the shock wave to be known; its co-ordinates are denoted by (η_s^*, ζ_s^*). As described in §3, there are two points of characteristic space corresponding to (η_s^*, ζ_s^*). These points are denoted by $(\eta_{0s}^*, \zeta_{0s}^*)$ and $(\hat{\eta}_{0s}^*, \hat{\zeta}_{0s}^*)$, corresponding to the state (u_1, v_1, w_1) in front of the shock and the state $(\hat{u}_1, \hat{v}_1, \hat{w}_1)$ behind the shock. In order to compute a neighbouring point (η_s, ζ_s) of the shock one has to compute the corresponding points (η_{0s}, ζ_{0s}) and $(\hat{\eta}_{0s}, \hat{\zeta}_{0s})$ at the same time. Thus six equations are necessary to compute a neighbouring point. These equations are

$$\left.\begin{aligned} \eta_s &= \eta_{0s} + \eta_1(\eta_{0s}, \zeta_{0s}), \quad & \zeta_s &= \zeta_{0s} + \zeta_1(\eta_{0s}, \zeta_{0s}), \\ \eta_s &= \hat{\eta}_{0s} + \eta_1(\hat{\eta}_{0s}, \hat{\zeta}_{0s}), \quad & \zeta_s &= \hat{\zeta}_{0s} + \zeta_1(\hat{\eta}_{0s}, \hat{\zeta}_{0s}) \end{aligned}\right\} \tag{10.9}$$

and two equations obtained from the shock conditions. The continuity of the velocity components tangential to the shock wave is in conical flow satisfied by the condition

$$\phi_1(\eta_{0s}, \zeta_{0s}) = \phi_1(\hat{\eta}_{0s}, \hat{\zeta}_{0s}). \tag{10.10}$$

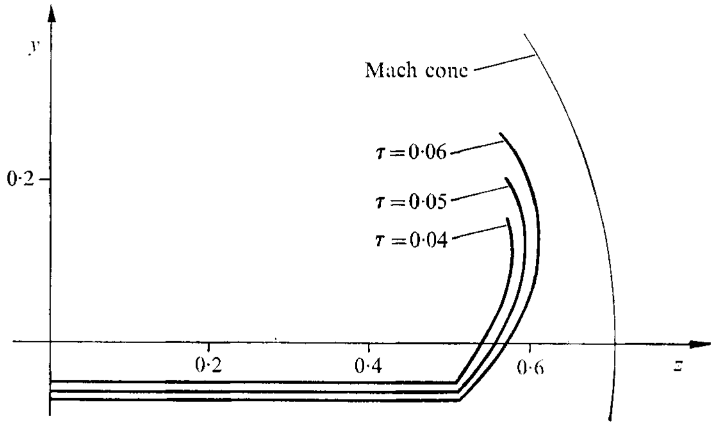

FIGURE 3. Location of shock waves. Mach number of undisturbed flow $M_0 = 2$, wing span $k/\beta = 0{\cdot}5$, angles of attack $\tau = 0{\cdot}04, 0{\cdot}05, 0{\cdot}06$ rad.

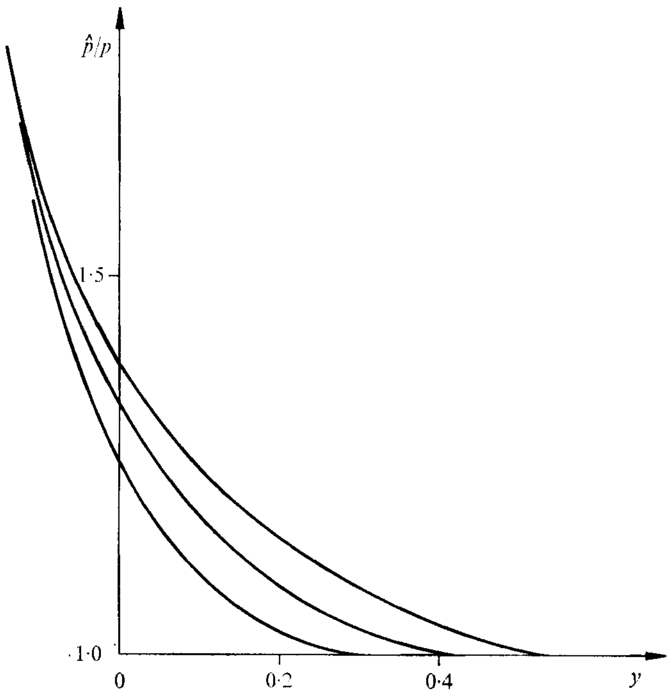

FIGURE 4. Pressure jump across shock waves. Mach number of undisturbed flow $M_0 = 2$, wing span $k/\beta = 0{\cdot}5$, angles of attack $\tau = 0{\cdot}04, 0{\cdot}05, 0{\cdot}06$ rad.

The jump condition normal to the shock is

$$u_n \hat{u}_n = c^{*2} - [(\kappa - 1)/(\kappa + 1)]\, u_t^2, \tag{10.11}$$

where u_n and $\hat{u}_n$ denote the velocity components normal to the shock and u_t the component tangential to the shock; c^* denotes the critical velocity of sound.

It is assumed that (10.9)–(10.11) can be expanded in the direction of the shock (not normal to it!) in terms of the small parameters

$$\eta_s^* - \eta_s, \quad \zeta_s^* - \zeta_s, \quad \eta_{0s}^* - \eta_{0s}, \quad \zeta_{0s}^* - \zeta_{0s}, \quad \hat{\eta}_{0s}^* - \hat{\eta}_{0s}, \quad \hat{\zeta}_{0s}^* - \hat{\zeta}_{0s}. \tag{10.12}$$

Thus six linear equations have to be solved for the six unknown values (10.12) This has been carried out numerically. Starting with the point at the leading edge, the location of the shock wave is computed up to its end-point. The shock strength is computed at the same time from the values of the velocity at the points (η_{0s}, ζ_{0s}) and $(\hat{\eta}_{0s}, \hat{\zeta}_{0s})$. At the end-point the shock strength vanishes.

Figure 2 shows the wing with the shock waves which arise at the leading edges and end at some distance from the wing, becoming infinitely weak at the end-points. Figure 3 shows the location of the shock waves for three different values of incidence. Figure 4 shows the pressure jump across these three shocks.

REFERENCES

Bulakh, B. M. 1970 Nichtlineare kegelige Strömungen. Moscow: Nauka.

Fraenkel, L. E. & Watson, K. 1964 The formulation of a uniform approximation for thin conical wings with sonic leading edges. *Symposium Transsonicum, Aachen*, 1962, p. **249**. Springer.

Guderley, G. 1954 The flow over a flat plate with a small angle of attack at Mach number 1. *J. Aero. Sci.* **21**, 261.

Lighthill, M. J. 1949 The shock strength in supersonic conical fields. *Phil. Mag.* **40**, 1202–1223.

Niederdrenk, P. 1969 Berechnung der Überschallströmung um einen angestellten Rotationskörper mit Hilfe eines analytischen Charakteristikenverfahrens. *DLR–FB*, no. 69–49.

Oswatitsch, K. 1962 Das Ausbreiten von Wellen endlicher Applitude *Z. Flugwiss.* **10**, H 4/5.

Robinson, A. 1946 Aerofoil theory of a flat delta wing at supersonic speeds. *R.A.E. Aero. Rep.* no. 2151.

Schneider, W. 1963 Analytische Berechnung achsensymmetrischer Überschallströmungen mit Stößen. *DLR-Bericht*, no. 275.

Stuff, R. 1968 Analytische Berechnung von Verdichtungsstößen beschleunigter oder verzögerter Rotationskörper. *DLR-FB*, no. 68–62.

Sun, E. Y. C. 1964 Nichtangestellte Deltaflügel mit Unterschallund Schallvorderkanten. *J. Mécanique*, **3**, 141–163.